CHOW RING AND BP-THEORY OF THE EXTRASPECIAL 2-GROUP OF ORDER 32

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ABSTRACT. We write down the mod 2 Chow ring of the classifying space of $G = 2^{1+4}_{+} = D_8 \cdot D_8$, which has nilpotent elements.

1. Introduction

Let p be a prime number. Let G be a p-group and BG its classifying space. Let us write simply by $H^*(G;\mathbb{Z}/p) = H^*(BG;\mathbb{Z}/p)$ the mod p cohomology of the group G, and by $CH^*(G) = CH^*(BG)$ the Chow ring of the classifying space BG over the complex number field $\mathbb{C}$.

In this paper, we write down the (most ease) case where $CH^*(G)/2$ has nonzero nilpotent elements (but $H^*(G;\mathbb{Z}/2)$ has not). Note that Chow rings $CH^*(G)/p$ for all G with $|G| \le p^4$ are still computed by Totaro in [To2]. Let $D(2) = 2^{1+4}_{+} = D_8 \cdot D_8$ be the extraspecial $2-$group (of order 2^5) which is the central product of two dihedral groups D_8.

Theorem 1.1. *There are ring isomorphisms*

$$CH^*(D(2))/2 \cong (H^*(D(2);\mathbb{Z}/2))^2 \oplus \mathbb{Z}/2[c_4]\{t''\}$$

$$\cong (\mathbb{Z}/2[y_1, y_2, y_3, y_4]/(q_0', q_1') \oplus \mathbb{Z}/2\{t''\}) \otimes \mathbb{Z}/2[c_4]$$

where $deg(y_i) = 1$, $deg(c_4) = 4$, $deg(t'') = 2$, and $q_0' = y_1y_2 + y_3y_4$,

$$q_1' = Sq^2(q_0') = y_1^2y_2 + y_1y_2^2 + y_3^2y_4 + y_3y_4^2.$$

The multiplications are given $(t'')^2 = y_it'' = 0$ for all $1 \le i \le 4$.

Let $BP^*(G) = BP^*(BG)$ be the Brown-Peterson theory with the coefficient $BP^* = \mathbb{Z}_{(p)}[v_1, v_2, ...]$ and $|v_i| = -2(p^i - 1)$ (for detalis of the BP-theory, see [Ha] or [Ra]). We also show the mod 2 Totaro conjecture ([To1]) ;

Theorem 1.2. *The mod 2 Totaro conjecture holds for $D(2)$, that is*

$$CH^*(D(2))/2 \cong BP^*(D(2)) \otimes_{BP^*} \mathbb{Z}/2.$$

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Let us write by $\Omega^*(G)$ the BP-version $\Omega^*(BG) \otimes_{MU^*} BP^*$ of the algebraic cobordism $\Omega^*(BG) = MGL^{2*,*}(BG)$ ([Vo1,2], [Le-Mo1,2]). Let $t_{\mathbb{C}} : \Omega^*(X) \to BP^{2*}(X(\mathbb{C}))$ be the realization map. There is a conjecture such that ;

Conjecture 1.3. *The realization map $t_{\mathbb{C}}$ is an isomorphism for each algebraic group G, e.g. $\Omega^*(BG) \cong BP^*(BG)$.*

It is known the above conjecture is true for connected groups ([To1], [Ya2,3]); $O_n,\ SO_n, PGL_p, G_2, Spin_7$. As for finite groups G, the above conjecture is known to be true for abelian groups and the extraspecial p-groups of order p^3, i.e. p_+^{1+2}, p_-^{1+2} for all primes [Ya4]. While the author can not see this conjecture for $D(2)$, in the last section, we add some notes for groups satisfying the above conjecture.

2. The Chow ring of $D(2)$

The group $D(2)$ is isomorphic to the extraspecial 2-group 2_+^{1+2}, which has the central extension

$$1 \to N \to D(2) \to Q \to 1, \quad N \cong \mathbb{Z}/2,\ Q \cong (\mathbb{Z}/2)^4.$$

We use notations such that $N \cong \langle c\rangle, Q \cong \langle a_1, a_2, a_3, a_4\rangle$ and

$$D(2) \cong \langle a_1, ..., a_4, c | a_1^2 = ... = a_4^2 = c^2 = 1,$$
$$[a_1, a_2] = [a_3, a_4] = c = (a_1a_2)^2 = (a_3a_4)^2\rangle.$$

The mod 2 cohomology is given by Quillen [Qu1]

$$H^*(D(2); \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, x_2, x_3, x_4]/(q_0, q_1) \otimes \mathbb{Z}/2[w_4]$$

where $q_0 = x_1x_2+x_3x_4$ and $q_1 = Sq^1q_0 = x_1^2x_2+x_1x_2^2+x_3^2x_4+x_3x_4^2$. Here x_i (and w_4) are Stiefel-Whitney classes for some real representations, and hence the powers are Chern classes, that is,

$$y_i = x_i^2 = c_1(e_i), \quad e_i : D(2) \to \langle a_i\rangle \to \mathbb{C}^\times$$

where e_i is the nonzero linear representation, and

$$c_4 = (w_4)^2 = c_4(\eta); \quad \eta = Ind_H^D(e),$$

where $H = \langle c, a_1, a_3\rangle$ is the maximal elementary abelian 2-subgroup of $D(2)$ and $e : H \to \langle c\rangle \to \mathbb{C}^\times$ is a nonzero linear representation. We note that $H^*(D(2); \mathbb{Z}/2)$ has no nonzero nilpotent elements ([Qu1]).

It is well known (e.g., [Qu1]) that each irreducible representation of an extraspecial p-group P is a linear representation or just one induced representation of a linear representation of a maximal elementary abelian p-group of P. Hence the Chern subring (the subring of $H^*(D(2); \mathbb{Z}/2)$ multiplicatively generated by Chern classes) is

$$Ch(H^*(D(2); \mathbb{Z}/2)) \cong H^*(D(2); \mathbb{Z}/2)^2$$

$$\cong \mathbb{Z}/2[y_1,...,y_4]/(q_0',q_1')\otimes \mathbb{Z}/2[c_4]$$

where $q_0' = y_1y_2+y_3y_4$ and $q_1' = Sq^2q_0' = y_1^2y_2+y_1y_2^2+y_3^2y_4+y_3y_4^2$.

Now we start to consider the Chow ring of $BD(2)$. In this paper we write $CH^*(BD(2))$ by $CH^*(D(2))$ (we also write $BP^*(BD(2))$ by $BP^*(D(2))$).

Moreover we note following facts (see [To1] for details). By the Rieman-Roch theorem without denominator, $CH^2(D(2))/2$ is generated by $2nd$ Chern classes (of some representations), that means, it is generated by y_iy_j and $c_2(\eta)$.

Lemma 2.1. *We have $q_0' = y_1y_2+y_3y_4 = 0 \in CH^2(D(2))/2$ and*

$$CH^2(D(2))/2 \cong \mathbb{Z}/2\{y_iy_j|1\le i,j\le 4\}/(q_0')\oplus \mathbb{Z}/2\{c_2(\eta)\}$$

where $A\{a,b,..\}$ means the free A-module generated by $a,b,....$

Proof. By Totaro (Corollary 3.5 in [To1] or Lemma 15.1 in [To2]), the integral cycle map

$$cl_{int}: CH^2(X)_{(2)} \to H^4(X;\mathbb{Z}_{(2)})$$

is injective. The higher 2-torsion of the integral cohomology of extraspecial 2-groups are studied by Harada-Kono ([Ha-Ko], [Sc-Ya1]). Let $C(2)^* = H^*(D(2))/J_Q$ where J_Q is the ideal generated by the image of $H^*(Q)$ in $H^*(D(2))$ (for $Q\cong(\mathbb{Z}/2)^4$). Then Harada-Kono show that

$$\mathbb{Z}/2^{s(*)} \cong C(2)^* \subset H^*(D(2)),$$

and when $* = 4m$, we have $C(2)^*\cong \mathbb{Z}/8$. Let w_4 be a generator of $C(2)^4$. Then it is known

$$w_4|N = u^2 \quad (w_4|N' = (u')^2 \quad for\ N' = \langle a_1a_2\rangle \cong \mathbb{Z}/4)$$

identifying $H^*(N)\cong \mathbb{Z}[u]/(2u)$ and $H^*(N')\cong\mathbb{Z}[u']/(4u')$.

On the other hand, all elements in J_Q are just 2-torsion. Moreover q_0' is (zero or) just 2-torsion (since so are y_i). Therefore we get

$$cl_{int}(q_0') = 4\lambda w_4 \quad for\ some\ \lambda\in\mathbb{Z}/8.$$

Let $c(\eta)=\sum c_i(\eta)$ is the total Chern class. Then we see

$$c(\eta)|_{N'} = (1+u')^4 = 1+4u'+6(u')^2+... \quad mod(8).$$

Hence $c_2(\eta)|_{N'} = -2(u')^2$ and so $q_0' = -2\lambda c_2(\eta)$. □

We recall a theorem of Totaro.

Theorem 2.2. *(Theorem 11.1 in [To2]) Let P be a p-group such that P has a faithful complex representation of dimension at most $p+2$. Then the mod p Chow ring of BP consists of transferred Euler classes.*

First note that Euler classes of $CH^*(D(2))$ are (multiplicatively) generated by $y_1, ..., y_4$ and $c_4(\eta)$. Next we consider the transfer images. Each proper maximum subgroup M of $D(2)$ is isomorphic to $D_8 \oplus \mathbb{Z}/2$, and let it be $\langle a_1, a_2, c, a_3\rangle$. The Chow ring $CH^*(M)/2$ is generated by Chern classes

$$y_1, y_2, y_3, \quad and \quad c_2 = c_2(\eta')$$

where $\eta' = Ind_H^M(e)$ and recall that $e : H = \langle c, a_1, a_3\rangle \to \mathbb{C}^\times$. Let us write the transfer $t_2 = Tr_M^{D(2)}(c_2)$. We note (by the double coset formula) $t_2|_{N'=\langle a_1a_2\rangle} = 2(u')^2$ identifying $CH^*(N') \cong \mathbb{Z}[u']/(4u')$ where $N' \cong \mathbb{Z}/4$. Therefore $t_2 = c_2(\eta)\ mod(y_iy_j)$ in $CH^*(D(2))/2$ from Lemma 2.1. Of course $Tr_M^{D(2)}(y_ic_2) = y_it_2$ for all $1 \leq i \leq 4$.

For an other proper maximal subgroup $\tilde{M}$, we similarly have the transfer $\tilde{t}_2$. However we have

$$\tilde{t}_2 = c_2(\eta) = t_2 \quad mod(y_iy_j).$$

From the Totaro theorem (Theorem 2.2), we have ;

Lemma 2.3. *The mod 2 Chow ring $CH^*(D(2))$ is multilpicatively generated by $y_1, ..., y_4$, $c_4 = c_4(\eta)$ and t_2 (or $c_2(\eta)$).*

Next we study the nilpotent elements. Let us write by cl the mod 2 cycle map

$$cl : CH^*(D(2))/2 \to H^*(D(2); \mathbb{Z}/2).$$

Recall that the Chern subring of $H^*(D(2); \mathbb{Z}/2)$ is generated by y_i and $c_4(\eta)$. Since t_2 is a Chern class, we can take $y \in \mathbb{Z}[y_1, ..., y_4]$ such that $cl(t_2) = y \in H^*(D(2); \mathbb{Z}/2)$.

Let $t'' = t_2 - y$ in $CH^*(D(2))$ so that $cl(t'') = 0$ and t'' is a (nonzero) nilpotent element in $CH^*(D(2))$ because $Ker(t_{\mathbb{C}})$ is nilpotent, since $t_{\mathbb{C}}$ is F-isomorphic from the Quillen theorem for Chow rings [Ya2]. (Note t'' is nonzero in $CH^*(D(2))/2$ because $t''|_{N'} = 2(u')^2$ and $CH^2(D(2))|_{N'}$ is generated by $2(u')^2$.)

Lemma 2.4. *$y_4t'' = 0$ in $CH^*(D(2))/2$.*

Proof. Note that

$$y_4t'' = y_4(t_2 - y) = tr_M^{D(2)}(y_4|_M \cdot c_2) - y_4y = -y_4y,$$

where we used $y_4|_M = 0$. Note y_4t'' is nilpotent but $H^*(D(2); \mathbb{Z}/2)$ has no nonzero nilpotent element. Hence $y_4y \in (q_0', q_1')$ and also zero in $CH^*(D(2))/2$. Thus $y_4t'' = 0$ in $CH^*(D(2))/2$. (Since $CH^*(X)$ has the reduced power operation Sq^2, we have $q_1' = Sq^2(q_0') = 0$ also in $CH^*(D(2))/2$ [Vo3].) □

Lemma 2.5. *For all $1 \leq i \leq 4$, we have $y_it'' = 0$.*

Proof. In $CH^2(D(2))/2$, nilpotent elements generate just one dimensional $\mathbb{Z}/2$-space $\mathbb{Z}/2\{t''\}$. Hence t'' is invariant under an action of the outer automorphism $Out(D(2))$. This outer automorphism contains

$$f : a_3 \leftrightarrow a_4, c \mapsto c, \qquad g : a_1 \mapsto a_3,\ a_2 \mapsto a_4,\ c \mapsto c.$$

We have $0 = f^*(y_4t'') = y_3t''$ and $0 = g^*(y_4t'') = y_2t''$. □

Lemma 2.6. *$(t'')^2 = 0$ in $CH^*(D(2))/2$.*

Proof. We compute

$$(t'')^2 = t''(tr_M^{D(2)}(c_2) - y) = t''tr_M^{D(2)}(c_2) = tr_M^{D(2)}(t''|_M \cdot c_2) = 0,$$

since $t''|_M$ is nilpotent but $CH^*(M)/2$ has no non zero nilpotent element. □

From the above lemmas, we get Theorem 1.1 in the introduction.

Remark. From Theorem in [To2], we see the topological nilpotency is $d_0(CH^*(D(2))/2) \leq 3$. This means $y_iy_jt'' = 0$. So we see a bit stronger result $d_0(CH^*(D(2))/2) = 2$ in the above lemma.

3. BP-theory

By Schuster-Yagita [Sc-Ya2], it is known that the Morava K-theory $K(n)^*(BD(2))$ is generated by even dimensional elements (see also Schuster [Sc] or Bakladze-Jibradze [Ba-Ji]) for all $n \geq 0$. This implies that $BP^*(D(2))$ is generated by even dimensional elements, and satisfies the condition of the Landweber exact functor theorem.

Moreover $D(2)$ is $K(n)^*$-good, namely, $K(n)^*(D(2))$ is generated by transferred Euler classes for all n. It is known ([Ra-Wi-Ya]) that it implies that $D(2)$ is BP^*-good, i.e., $BP^*(D(2)$ is generated also by transferred Euler classes.

Recall the exact sequence

$$(*) \quad 0 \to M \cong D_8 \oplus \mathbb{Z}/2 \to D(2) \to \mathbb{Z}/2 \to 0$$

Here we use notations $M = \langle a_1, a_2, c, a_3\rangle$ and $\mathbb{Z}/2 \cong \langle a_4\rangle$ in the following proof.

Proof of Theorem 1.2. The cycle map is decomposed as

$$cl : CH^*(X)/2 \overset{cl_{BP}}{\to} BP^*(X) \otimes_{BP^*} \mathbb{Z}/2 \overset{\rho}{\to} H^*(X;\mathbb{Z}/2)$$

where cl_{BP} is the Totaro cyle map and ρ is the Thom map.

By the BP^*-goodness of $D(2)$, we see that cl_{BP} is surjective. Moreover it is known ([Ya2]) that cl_{BP} is an F-isomorphism. Hence $Ker(cl_{BP})$ is nilpotent. Thus it is only need to show

$$\mathbb{Z}/2[c_4]\{t''\} \subset BP^*(D(2)) \otimes_{BP^*} \mathbb{Z}/2.$$

(Note that t'' exists in $BP^*(D(2)$, but we need to see $t'' \neq 0$ and t'' generates a $\mathbb{Z}/2[c_4]$-free module.)

Note $t''|_{N'} = 2(u')^2$ and so $t''|_M$ is not a BP^*-module generator of $BP^*(M)$ but $c_2(\eta') \notin BP^*(M)^{\langle a_4\rangle}$. Hence $t''|_M$ is a BP^*-module generator of $BP^*(M)^{\langle a_4\rangle}$. Then we have

$$BP^*(D(2)) \otimes_{BP^*} \mathbb{Z}/2 \stackrel{res}{\to} BP^*(M)^{\langle a_4\rangle} \otimes_{BP^*} \mathbb{Z}/2 \supset \mathbb{Z}/2[c_4]\{t_2''|_M\}.$$

The last inclusion follows from the restriction to $N' = \langle a_1 a_2\rangle \cong \mathbb{Z}/4$,

$$BP^*[c_4](t'')|_{N'} = BP^*[(u')^4]\{2u'\} \subset BP^*(N') \cong BP^*[u']([4](u')).$$

Thus we have the theorem. □

In this paper, we do not explicitly use the following lemma and corollary, but we note them.

Lemma 3.1. *The restriction map*

$$res : BP^*(G) \to Lim_{G\supset A:abelian} BP^*(A)$$

is an F-isomorphism (i.e., its kernel and cokernel are nilpotent).

Proof. We can define the Evens norm for BP^*-theory. Hence res is F-surjective from the arguments in the proof of Lemma 2.4 in [Qu2]. The F-injective follows from the arguments (3.10) in page 371 in [Qu2]. □

Note that A ranges all abelian subgroups of G for the F-injectivity. In fact, the kernel of $BP^*(\mathbb{Z}/4) \cong BP^*[u']/([4](u')) \to BP^*(\mathbb{Z}/2)$ is the ideal $[2](u')$ which is not nilpotent.

Corollary 3.2. *$BP^*(D(2)) \subset Lim_{D(2)\supset A:abel.} BP^*(A)$.*

4. Algebraic cobordism $\Omega^*(P)$

Let p be a fixed prime number. For a smooth variety X over the complex field $\mathbb{C}$, let us write by

$$\Omega^*(X) = MGL^{2*,*}(X) \otimes_{MU^*} BP^* \cong ABP^{2*,*}(X)$$

the (BP^*-version of) algebraic cobordism defined by Voevodsky ([Vo1,2]) and Levine-Morel ([Le-Mo1,2]). There is a conjecture (Conjecture 1.3) such that the realization map induces the isomorphism $t_{\mathbb{C}} : \Omega^*(BG) \cong BP^*(BG)$ for the classifying space BG of each algebraic group G.

It is known that this conjecture is true for connected groups [Ya2,3] $O_n,\ SO_n, PGL_p, G_2, Spin_7$. As for finite groups G, it is known that the conjecture is true for abelian groups and the extraspecial p-groups p_+^{1+2}, p_-^{1+2} for all primes [Ya4]. In this section, we show the conjecture for other p-groups.

We consider a p-group G and its subgroup M of index p^s, namely, there is the extension

$$(*) \quad 1 \to M \to G \to \mathbb{Z}/p^s \to 0$$

and consider the induced spectral sequence

$$E_2^{*,*'} \cong H^*(\mathbb{Z}/p^s; BP^*(M)) \Longrightarrow BP^*(G).$$

Let the right hand side group $\mathbb{Z}/p^s$ in $(*)$ be generated by a. Let $N = 1 + a^* + ... + (a^{p^s-1})^*$ and recall that

$$E_2^{*,*'} \cong \begin{cases} Ker(1-a^*) \cong BP^*(M)^{\langle a \rangle} & * = 0 \\ Ker(1-a^*)/Im(N) & * = even > 0 \\ KerN/Im(1-a^*) & * = odd. \end{cases}$$

We consider the cases that $E_2^{odd,*'} \cong 0$.

Lemma 4.1. *Let G be a p-group with the extension $(*)$ such that $E_2^{odd,*'} = 0$. Moreover we assume ;*

(1) The mod(p) Totaro conjecture holds for G, i.e.

$$CH^*(G)/p \cong BP^*(G) \otimes_{BP^*} \mathbb{Z}/p,$$

(2) The conjecture 1.3 holds for M, i.e. $t_{\mathbb{C}} : \Omega^*(M) \cong BP^*(M)$.

Then Conjecture 1.3 holds for G, namely, $t_{\mathbb{C}} : \Omega^*(G) \cong BP^*(G)$.

Proof. Let y be the first Chern class of a nonzero linear representation for G ; $G \to \langle a \rangle \to \mathbb{C}^*$. Then from $E_2^{odd,*'} \cong 0$, we see

$$E_\infty^{*,*'} \cong E_\infty^{even,*'} \cong E_2^{even,*'}.$$

Hence we get

$$grBP^*(G) \cong BP^*(M)^{\langle a \rangle} \oplus (BP^*(M)^{\langle a \rangle}/N)[y]^+.$$

On the other hand, from (1), the algebraic cobordism $\Omega^*(G)$ is also generated by $BP^*(M)^{\langle a \rangle} (\cong \Omega^*(M)^{\langle a \rangle})$ and $y \in \Omega^2(G)$. We consider the filtration defined by the $ideal(y) \subset \Omega^*(M)$.

For $x \in \Omega(M)^{\langle a \rangle}$, take $\tilde{x} \in \Omega^*(G)$ with $\tilde{x}|_M = x$ (which is only decided with modulo $Ideal(y)$). (Note we can take $\tilde{N}x = Tr_M^G(x)$.) Then $\Omega^*(G)$ is additively generated by $\tilde{x}$ and $\tilde{x}y^i$. Hence we have

$$gr\Omega^*(G) \cong BP^*(M)^{\langle a \rangle} \oplus \oplus_{i \geq 1} (BP^*(M)^{\langle a \rangle}/N_i)\{y^i\}$$

where $N_1 \subset N_2 \subset ....$ Note $N_i \subset Im(N)$, since we have the cycle map $gr\Omega^*(G) \to grBP^*(G)$. Hence we only need to prove $N_1 = Im(N)$.

For $x \in \Omega^*(M)$, we see

$$y\tilde{N}(x) = yTr_M^G(x) = Tr_M^G((y|_M) \cdot x) = 0 \quad in \ \Omega^*(G).$$

Thus $Im(N) \subset N_1$ and we see $N_i = Im(N)$ for all i. $\square$

For $G = D(2)$, we consider the exact sequence $(*)$ in §3, and the induced spectral sequence converging to $BP^*(D(2))$.

Corollary 4.2. *If $E_2^{odd,*'} = 0$ for the above spectral sequence, then Cojecture 1.3 holds for $D(2)$.*

Next we consider groups P with $rank_p = 2$ and $p \geq 3$. At first, we consider a split metacyclic group. It is written

$$P = M(\ell, m, n) = \langle a, b | a^{p^m} = b^{p^n} = 1, [a, b] = a^{p^\ell} \rangle$$

for $m > \ell \geq max(m - n, 1)$. Consider the extension

$$1 \to \langle a \rangle \to P \to \langle b \rangle \to 1.$$

Then this extension satisfies the assumption in Lemma 4.1 except for (1) ([Te-Ya2]) and $BP^*(P) \otimes_{BP^*} \mathbb{Z}_{(p)} \cong H^{even}(P; \mathbb{Z}_{(p)})$. Moreover when $m - \ell = 1$, Totaro showed the above cohomology is isomorphic to the Chow ring $CH^*(P)$ [To2]. Therefore we have

Corollary 4.3. *Conjecture 1.3 holds for $M(m, \ell, n)$ with $m - \ell = 1$.*

We consider the other $rank_p P = 2$ groups. For $p \geq 5$, groups P with $rank_p P = 2$ are classified by Blackburn (see [Ya1]). They are metacyclic groups, and some groups $C(r)$, $G(r', e)$. The group $C(r)$, $r \geq 3$ is defined by

$$C(r) = \langle a, b, a | a^p = b^p = c^{p^{r-2}} = 1, \ [a, b] = c^{p^{r-3}} \rangle$$

for $r \geq 3$ so that $C(3) = p_+^{1+2}$. The group $G = G(r, e), r \geq 4$ (and $e \neq 0$ is a quadratic nonresidue mod p) is defined as

$$\langle a, b, c | a^p = b^p = c^{p^{r-2}} = [b, c] = 1, [a, b^{-1}] = c^{ep^{r-3}}, [a, c] = b \rangle.$$

The subgroup $\langle a, b, c^p \rangle$ is isomorphic to $C(r - 1)$.

Corollary 4.4. *Conjecture 1.3 holds for $C(r), D(r + 1, e)$.*

Proof. It is known $CH^*(P)/p \cong H^{even}(P; \mathbb{Z})/p \cong BP^*(P) \otimes_{BP^*} \mathbb{Z}/p$. Here the first isomorphism is proved in [To2] and the second is shown in [Ya1]. The extension

$$1 \to \langle c, a \rangle \to (r) \to \langle b \rangle \to 1$$

satisfies [Ya1] the assumption Lemma 4.1 for $C(r)$ The extension

$$1 \to \langle a, b, c^p \rangle \to G(r + 1, e) \to \langle c \rangle \to 1$$

satisfies [Ya1] the assumption of Lemma 4.1 for $G(r + 1, e)$. □

We write down the result for p-Sylow subgroups $\mathbb{Z}/p \wr ... \wr \mathbb{Z}/p$ of symmetric groups. Here $\mathbb{Z}/p \wr X = \mathbb{Z}/p \rtimes (X)^{\times p}$ is the p-th wreath product.

Corollary 4.5. *Conjecture 1.3 holds for $\mathbb{Z}/p \wr ... \wr \mathbb{Z}/p$.*

Proof. Totaro's conjecture is still proved in [To1]. We consider the extension

$$1 \to (G')^p \to \mathbb{Z}/p \wr G' \to \mathbb{Z}/p \to 1$$

and induced spectral sequence converging to $BP^*(\mathbb{Z}/p \wr G')$. It is proved in Lemma 5.3 in [Te-Ya2] that if there exist BP^*-module generators $\{x_i\}$ of $BP^*(G')$ such that $\{\rho(x_i)\}$ is a subset of $\mathbb{Z}/p$-basis of $H^*(G')/p$, then $E_2^{odd,*'} = 0$. By induction on the number of the wreath product, we can show the corollary. □

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